

LOWER BOUNDS FOR THE SPECTRUM OF THE LAPLACE AND STOKES OPERATORS

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ABSTRACT. We prove Berezin–Li–Yau-type lower bounds with additional term for the eigenvalues of the Stokes operator and improve the previously known estimates for the Laplace operator. Generalizations to higher-order operators are given.

Dedicated to Professor R. Temam on the occasion of his 70th birthday

1. INTRODUCTION

Sharp lower bounds for the sums of the first m eigenvalues of the Dirichlet Laplacian

$$-\Delta\varphi_k = \mu_k\varphi_k, \quad \varphi_k|_{\partial\Omega} = 0$$

were obtained in [10]:

$$\sum_{k=1}^m \mu_k \geq \frac{n}{2+n} \left(\frac{(2\pi)^n}{\omega_n|\Omega|} \right)^{2/n} m^{1+2/n}. \quad (1.1)$$

Here $|\Omega| < \infty$ denotes the volume of a domain $\Omega \subset \mathbb{R}^n$ and ω_n denotes the volume of the unit ball in \mathbb{R}^n . It was shown in [9] that the estimate (1.1) is equivalent by means of the Legendre transform to an earlier result of Berezin [3].

In view of the classical H. Weyl asymptotic formula

$$\mu_k \sim \left(\frac{(2\pi)^n}{\omega_n|\Omega|} \right)^{2/n} k^{2/n} \quad \text{as } k \rightarrow \infty,$$

the coefficient of $m^{1+2/n}$ in (1.1) is sharp. However, an improvement of the Li–Yau bound with additional term that is linear in m was obtained in [11]:

$$\sum_{k=1}^m \mu_k \geq \frac{n}{2+n} \left(\frac{(2\pi)^n}{\omega_n|\Omega|} \right)^{2/n} m^{1+2/n} + c_n \frac{|\Omega|}{I} m, \quad (1.2)$$

where

$$I = \int_{\Omega} x^2 dx, \quad (1.3)$$

and the constant c_n depends only on the dimension: $c_n = c/(n+2)$ with c being an absolute constant (in fact, (1.2) holds with $c = 1/24$). Of course I can be replaced by $I = \min_{a \in \mathbb{R}^n} \int_{\Omega} (x-a)^2 dx$.

In the theory of the attractors for the Navier–Stokes equations (see, for example, [2, 4, 15] and the references therein) lower bounds for the sums of the eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ of the

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Stokes operator are very important. In the case of a smooth domain the eigenvalue problem for the Stokes operator reads:

$$\begin{aligned} -\Delta v_k + \nabla p_k &= \lambda_k v_k, \\ \operatorname{div} v_k &= 0, \quad v_k|_{\partial\Omega} = 0. \end{aligned} \quad (1.4)$$

Li–Yau-type lower bounds for the spectrum of the Stokes operator were obtained in [6]:

$$\sum_{k=1}^m \lambda_k \geq \frac{n}{2+n} \left(\frac{(2\pi)^n}{\omega_n(n-1)|\Omega|} \right)^{2/n} m^{1+2/n}. \quad (1.5)$$

The coefficient of $m^{1+2/n}$ here is also sharp in view of the asymptotic formula ([1] ($n=3$), [12] ($n \geq 2$)):

$$\lambda_k \sim \left(\frac{(2\pi)^n}{\omega_n(n-1)|\Omega|} \right)^{2/n} k^{2/n} \quad \text{as } k \rightarrow \infty. \quad (1.6)$$

The main results of this paper are twofold. First, we extend the approach of [11] to the case of the Stokes operator and, secondly, we obtain the exact solution of the corresponding minimization problem, thereby giving a much better value of the constant c_n in (1.2) (in fact, the sharp value in the framework of the approach of [11]).

To describe the minimization problem we consider in the case of the Laplacian an L_2 -orthonormal family of functions $\{\varphi_k\}_{k=1}^m \in H_0^1(\Omega)$. Then the function $F(\xi)$

$$F(\xi) = \sum_{k=1}^m |\widehat{\varphi}_k(\xi)|^2, \quad \widehat{\varphi}_k(\xi) = (2\pi)^{-n/2} \int_{\Omega} \varphi_k(x) e^{-i\xi x} dx$$

satisfies $F(\xi) \leq M = (2\pi)^{-n}|\Omega|$ (see [10]) and the additional regularity property which was found and used in [11]: $|\nabla F(\xi)| \leq L = 2(2\pi)^{-n}\sqrt{|\Omega|I}$. Here and in what follows I is defined in (1.3).

For the Stokes operator we consider an \mathbf{L}_2 -orthonormal family of vector functions $\{u_k\}_{k=1}^m \in \mathbf{H}_0^1(\Omega)$, with $\operatorname{div} u_k = 0$. Then as we show in §2 the corresponding function $F(\xi) = \sum_{k=1}^m |\widehat{u}_k(\xi)|^2$ satisfies the conditions $F(\xi) \leq M = (2\pi)^{-n}(n-1)|\Omega|$ and $|\nabla F(\xi)| \leq L = 2(2\pi)^{-n}(n(n-1))^{1/2}\sqrt{|\Omega|I}$.

By orthonormality we always have $\int_{\mathbb{R}^n} F(\xi) d\xi = m$, and taking the first m eigenfunctions of the Laplace (or Stokes) operator for the φ_k (or the u_k , respectively) we get $\int_{\mathbb{R}^n} |\xi|^2 F(\xi) d\xi = \sum_{k=1}^m \mu_k$ ($= \sum_{k=1}^m \lambda_k$), and $\sum_{k=1}^m \mu_k \geq \Sigma_M(m)$ (respectively, $\sum_{k=1}^m \lambda_k \geq \Sigma_M(m)$), where $\Sigma_M(m)$ is the solution of the minimization problem: find $\Sigma_M(m)$

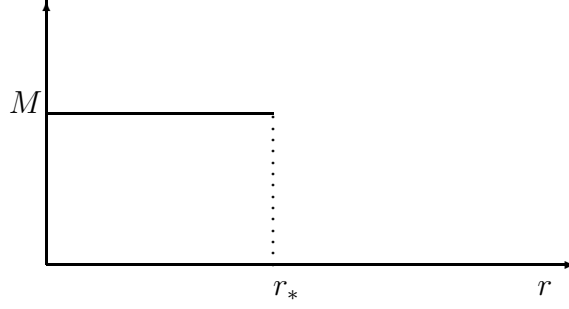
$$\begin{aligned} \int_{\mathbb{R}^n} |\xi|^2 F(\xi) d\xi &\rightarrow \inf := \Sigma_M(m), \quad \text{under the conditions} \\ 0 \leq F(\xi) &\leq M, \quad \int_{\mathbb{R}^n} F(\xi) d\xi = m. \end{aligned} \quad (1.7)$$

It was shown in [10] that the minimizer F_* is radial and has the form shown in Fig. 1, where r_* is defined by the condition $\int_{\mathbb{R}^n} F_*(|\xi|) d\xi = m$:

$$\int_{\mathbb{R}^n} F_*(|\xi|) d\xi = \sigma_n \int_0^{r_*} r^{n-1} F_*(r) dr = \omega_n M r_*^n = m.$$

Then

$$\Sigma_M(m) = \int_{\mathbb{R}^n} |\xi|^2 F_*(\xi) d\xi = \sigma_n M \int_0^{r_*} r^{n+1} dr = \frac{n}{n+2} \left(\frac{1}{\omega_n M} \right)^{2/n} m^{1+2/n}$$


 FIGURE 1. Minimizer $F_*(|\xi|)$

giving (1.1) upon substituting $M = (2\pi)^{-n}|\Omega|$ for the Laplacian and giving (1.5) upon substituting $M = (2\pi)^{-n}(n-1)|\Omega|$ for the Stokes operator [6].

The additional regularity property of $F(\xi)$: $|\nabla F(\xi)| \leq L$ gives a better lower bound [11]: $\sum_{k=1}^m \mu_k \geq \Sigma_{M,L}(m)$, where $\Sigma_{M,L}(m)$ is the solution of the minimization problem

$$\begin{aligned} \int_{\mathbb{R}^n} |\xi|^2 F(\xi) d\xi \rightarrow \inf =: \Sigma_{M,L}(m) \quad \text{under the conditions,} \\ 0 \leq F(\xi) \leq M, \quad \int_{\mathbb{R}^n} F(\xi) d\xi = m, \quad |\nabla F(\xi)| \leq L. \end{aligned} \quad (1.8)$$

Clearly $\Sigma_{M,L}(m) \geq \Sigma_M(m)$ and Lemma 1 in [11] (in the notation our paper) reads:

$$\Sigma_{M,L}(m) \geq \frac{n}{n+2} \left(\frac{1}{\omega_n M} \right)^{2/n} m^{1+2/n} + \frac{1}{6(n+2)} \frac{M^2}{L^2} m, \quad (1.9)$$

giving (1.2) with $c_n = 1/(24(n+2))$ by substituting $M = (2\pi)^{-n}|\Omega|$ and $L = 2(2\pi)^{-n}\sqrt{|\Omega|I}$.

In §3 we find the exact solution of the minimization problem (1.8):

$$\Sigma_{M,L}(m) = \frac{\sigma_n M^{n+3}}{(n+2)(n+3)L^{n+2}} ((t(m_*) + 1)^{n+3} - t(m_*)^{n+3}),$$

where $t(m_*)$ is the unique positive root of the equation

$$(t+1)^{n+1} - t^{n+1} = m_*, \quad m_* = m \frac{(n+1)L^n}{\omega_n M^{n+1}}.$$

We also find the first three terms of the asymptotic expansion of the solution $\Sigma_{M,L}(m)$ in the following descending powers of m : $m^{1+2/n}$, m , $m^{1-2/n}$, $m^{1-4/n}$, \dots . Namely,

$$\begin{aligned} \Sigma_{M,L}(m) &= \Sigma_0(m) + O(m^{1-4/n}), \\ \Sigma_0(m) &= \frac{n}{n+2} \left(\frac{1}{\omega_n M} \right)^{2/n} m^{1+2/n} + \frac{n}{12} \frac{M^2}{L^2} m - \frac{n(n-1)(3n+2)}{1440} \frac{M^4 (M\omega_n)^{2/n}}{L^4} m^{1-2/n}, \end{aligned} \quad (1.10)$$

which shows that the second term is for all n linear with respect to m and positive with coefficient that is $n(n+2)/2$ times greater than that in (1.9), while the third term is always negative.

Dropping the third term and using the expressions for M and L we obtain the following asymptotic lower bounds. Accordingly, for large m the coefficient of m in the second term on the right-hand side in (1.11) is $n(n+2)/2$ times greater than that in (1.2).

Theorem 1.1. *The eigenvalues $\{\mu_k\}_{k=1}^\infty$ and $\{\lambda_k\}_{k=1}^\infty$ of the Laplace and Stokes operators satisfy the following lower bounds:*

$$\sum_{k=1}^m \mu_k \geq \frac{n}{n+2} \left(\frac{(2\pi)^n}{\omega_n |\Omega|} \right)^{2/n} m^{1+2/n} + \frac{n}{48} \frac{|\Omega|}{I} m (1 - \varepsilon_n(m)), \quad (1.11)$$

$$\sum_{k=1}^m \lambda_k \geq \frac{n}{n+2} \left(\frac{(2\pi)^n}{\omega_n (n-1) |\Omega|} \right)^{2/n} m^{1+2/n} + \frac{(n-1)}{48} \frac{|\Omega|}{I} m (1 - \varepsilon_n(m)), \quad (1.12)$$

where $\varepsilon_n(m) \geq 0$, $\varepsilon_n(m) = O(m^{-2/n})$.

Then in §4 we turn to the analysis of the particular cases $n = 2, 3, 4$. The main result consists in the explicit formulas for $\Sigma_{M,L}(m)$. The case $n = 2$ is the simplest and we find (see Lemma 4.1) the explicit formula for the exact solution which coincides with the first three terms of its asymptotic expansion

$$\Sigma_{M,L}(m) = \Sigma_0(m) = \frac{1}{2\pi M} m^2 + \frac{M^2}{6L^2} m - \frac{\pi M^5}{90L^4}.$$

For $n = 3, 4$ by means of the explicit formulas in Lemmas 4.3 and 4.2 we show that

$$\Sigma_{M,L}(m) > \Sigma_0(m).$$

(The inequality $\Sigma_{M,L}(m) \geq \Sigma_0(m)$ probably holds for any n , not only for $n = 2, 3, 4$.) Then the negative contribution from the third term in (1.10) is compensated by a $(1 - \beta)$ -part of the positive second term (where $0 < \beta < 1$ and β is sufficiently close to 1) and we obtain the following theorem.

Theorem 1.2. *The eigenvalues $\{\mu_k\}_{k=1}^\infty$ and $\{\lambda_k\}_{k=1}^\infty$ for $n = 2, 3, 4$ satisfy:*

$$\sum_{k=1}^m \mu_k \geq \frac{n}{n+2} \left(\frac{(2\pi)^n}{\omega_n |\Omega|} \right)^{2/n} m^{1+2/n} + \frac{n}{48} \beta_n^L \frac{|\Omega|}{I} m, \quad (1.13)$$

$$\sum_{k=1}^m \lambda_k \geq \frac{n}{n+2} \left(\frac{(2\pi)^n}{\omega_n (n-1) |\Omega|} \right)^{2/n} m^{1+2/n} + \frac{(n-1)}{48} \beta_n^S \frac{|\Omega|}{I} m, \quad (1.14)$$

where in the two-dimensional case $\beta_2^L = \frac{119}{120}$, $\beta_2^S = \frac{239}{240}$, while for $n = 3, 4$ it suffices to take $\beta_3^L = 0.986$, $\beta_3^S = 0.986$ and $\beta_4^L = 0.983$, $\beta_4^S = 0.978$.

Finally, in §5 we prove two-term lower bounds for the Dirichlet bi-Laplacian.

Remark 1.1. Two term lower bounds for the 2D Laplacian with the second term of growth higher than linear in m were obtained in [7]. They depend on the shape of $\partial\Omega$.

2. ESTIMATES FOR ORTHONORMAL VECTOR FUNCTIONS

Throughout Ω is an open subset of \mathbb{R}^n with finite n -dimensional Lebesgue measure $|\Omega|$:

$$\Omega \subset \mathbb{R}^n, \quad n \geq 2, \quad |\Omega| < \infty.$$

We recall the functional definition of the Stokes operator [4, 8, 14]: \mathcal{V} denotes the set of smooth divergence-free vector functions with compact supports

$$\mathcal{V} = \{u : \Omega \rightarrow \mathbb{R}^n, \quad u \in \mathbf{C}_0^\infty(\Omega), \quad \operatorname{div} u = 0\}$$

and H and V are the closures of \mathcal{V} in $\mathbf{L}_2(\Omega)$ and $\mathbf{H}^1(\Omega)$, respectively. The Helmholtz–Leray orthogonal projection P maps $\mathbf{L}_2(\Omega)$ onto H , $P : \mathbf{L}_2(\Omega) \rightarrow H$. We have (see [14])

$$\begin{aligned} \mathbf{L}_2(\Omega) &= H \oplus H^\perp, \quad H^\perp = \{u \in \mathbf{L}_2(\Omega), u = \nabla p, p \in L_2^{\text{loc}}(\Omega)\}, \\ V &\subseteq \{u \in \mathbf{H}_0^1(\Omega), \operatorname{div} u = 0\}, \end{aligned} \quad (2.1)$$

where the last inclusion becomes equality for a bounded Ω with Lipschitz boundary. The Stokes operator A is defined by the relation

$$(Au, v) = (\nabla u, \nabla v) \quad \text{for all } u, v \text{ in } V \quad (2.2)$$

and is an isomorphism between V and V' . For a sufficiently smooth u

$$Au = -P\Delta u.$$

The Stokes operator A is an unbounded self-adjoint positive operator in H with discrete spectrum $\{\lambda_k\}_{k=1}^\infty$, $\lambda_k \rightarrow \infty$ as $k \rightarrow \infty$:

$$Av_k = \lambda_k v_k, \quad 0 < \lambda_1 \leq \lambda_2 \leq \dots, \quad (2.3)$$

where $\{v_k\}_{k=1}^\infty \in V$ are the corresponding orthonormal eigenvectors. Taking the scalar product with v_k we have by orthonormality and (2.2) that

$$\lambda_k = \|\nabla v_k\|^2. \quad (2.4)$$

In case when Ω is a bounded domain with smooth boundary the eigenvalue problem (2.3) goes over to (1.4).

We recall that a family $\{\varphi_i\}_{i=1}^m \in L_2(\Omega)$ is called suborthonormal [5] if for any $\zeta \in \mathbb{C}^m$

$$\sum_{i,j=1}^m \zeta_i \zeta_j^* (\varphi_i, \varphi_j) \leq \sum_{j=1}^m |\zeta_j|^2. \quad (2.5)$$

Lemma 2.1. *Any suborthonormal family $\{\varphi_i\}_{i=1}^m$ satisfies Bessel's inequality:*

$$\sum_{k=1}^m c_k^2 \leq \|f\|_{L_2(\Omega)}^2, \quad \text{where } c_k = (\varphi_k, f). \quad (2.6)$$

Proof. Given an suborthonormal system $\{\varphi_i\}_{i=1}^m$ (with supports in Ω), we build it up to a orthonormal system $\{\psi_i\}_{i=1}^m \in L_2(\mathbb{R}^n)$ of the form $\psi_k = \varphi_k + \chi_k$, $\chi_k(x) = \sum_{j=1}^m a_{kj} \omega_j(x)$, where $\{\omega_i\}_{i=1}^m$ is an arbitrary orthonormal system with supports in $\mathbb{R}^n \setminus \Omega$. The condition $(\psi_k, \psi_l) = \delta_{kl}$ is satisfied if we chose for the matrix $a = a_{ij}$ the symmetric non-negative matrix $a = b^{1/2}$, where $b_{ij} = \delta_{ij} - (\varphi_i, \varphi_j)$ (in view of (2.5), b is non-negative).

The system $\{\psi_i\}_{i=1}^m$ classically satisfies Bessel's inequality, and since $(\psi_k, f) = (\varphi_k, f)$, this gives (2.6). \square

Suborthonormal families typically arise as a result of the action of an orthogonal projection [5].

Lemma 2.2. *If $\{\varphi_k\}_{k=1}^m$ is orthonormal and P is an orthogonal projection, then both families $\eta_k = P\varphi_k$ and $\xi_k = (I - P)\varphi_k$ are suborthonormal.*

We now obtain some estimates for the Fourier transforms for (sub)orthonormal families.

Lemma 2.3. *If $\{\varphi_k\}_{k=1}^m$ is suborthonormal, then*

$$\sum_{k=1}^m |\widehat{\varphi}_k(\xi)|^2 \leq (2\pi)^{-n} |\Omega|. \quad (2.7)$$

Proof. This follows from (2.6) with $f(x) = f_\xi(x) = (2\pi)^{-n/2} e^{-i\xi x}$. \square

Corollary 2.1. *If the family of vector functions $\{u_k\}_{k=1}^m$ is orthonormal in $\mathbf{L}_2(\Omega)$, then*

$$\sum_{k=1}^m |\widehat{u}_k(\xi)|^2 \leq (2\pi)^{-n} n |\Omega|. \quad (2.8)$$

Proof. By Lemma 2.2 for each $j = 1, \dots, n$ the family $\{u_k^j\}_{k=1}^m$ is suborthonormal and (2.8) follows from Lemma 2.3. \square

The next lemma [6] is essential for the Li–Yau bounds for the Stokes operator and says that under the additional condition $\operatorname{div} u_k = 0$ the factor n in the previous estimate is replaced by $n - 1$.

Lemma 2.4. *If the family of vector functions $\{u_k\}_{k=1}^m$ is orthonormal and $u_k \in H$, then*

$$\sum_{k=1}^m |\widehat{u}_k(\xi)|^2 \leq (2\pi)^{-n} (n - 1) |\Omega|. \quad (2.9)$$

Proof. First we observe that $\xi \cdot \widehat{u}_k(\xi) = (2\pi)^{-n/2} i \int u_k \cdot \nabla_x e^{-i\xi x} dx = 0$ for all $\xi \in \mathbb{R}_\xi^n$ since the u_k 's are orthogonal to gradients (see (2.1)). Let $\xi_0 \neq 0$ be of the form:

$$\xi_0 = (a, 0, \dots, 0), \quad a \neq 0. \quad (2.10)$$

Since $\xi_0 \cdot \widehat{u}_k(\xi_0) = 0$, it follows that $\widehat{u}_k^1(\xi_0) = 0$ for $k = 1, \dots, m$. Hence, by (2.7)

$$\sum_{k=1}^m |\widehat{u}_k(\xi_0)|^2 = \sum_{j=2}^n \sum_{k=1}^m |\widehat{u}_k^j(\xi_0)|^2 \leq (2\pi)^{-n} (n - 1) |\Omega|.$$

The general case reduces to the case (2.10) by the corresponding rotation of \mathbb{R}^n about the origin represented by the orthogonal $(n \times n)$ -matrix ρ . Given a vector function $u(x) = (u^1(x), \dots, u^n(x))$ we consider the vector function

$$u_\rho(x) = \rho u(\rho^{-1}x), \quad x \in \rho\Omega.$$

A straight forward calculation gives that $\operatorname{div} u_\rho(x) = \operatorname{div} u(y)$, where $\rho^{-1}x = y$. In addition, $(u_\rho, v_\rho) = (u, v)$. Combining this we obtain that the family $\{(u_k)_\rho\}_{k=1}^m$ is orthonormal and belongs to $H(\rho\Omega)$.

Next we calculate \widehat{u}_ρ and see that $\widehat{u}_\rho(\xi) = \rho \widehat{u}(\rho^{-1}\xi)$. We now fix an arbitrary $\xi \in \mathbb{R}^n$, $\xi \neq 0$ and set $\xi_0 = (|\xi|, 0, \dots, 0)$. Let ρ be the rotation such that $\xi = \rho^{-1}\xi_0$. Then we have

$$\sum_{k=1}^m |\widehat{u}_k(\xi)|^2 = \sum_{k=1}^m |\widehat{u}_k(\rho^{-1}\xi_0)|^2 = \sum_{k=1}^m |\rho^{-1}(\widehat{u_k})_\rho(\xi_0)|^2 = \sum_{k=1}^m |(\widehat{u_k})_\rho(\xi_0)|^2 \leq (2\pi)^{-n} (n - 1) |\Omega|,$$

where we have used that inequality (2.9) has been proved for ξ of the form (2.10) for any orthonormal family of divergence-free vector functions. Finally, the estimate (2.9) is extended to $\xi = 0$ by continuity. \square

For the orthonormal family $\{u_k\}_{k=1}^m \in H$ we set

$$F_S(\xi) = \sum_{k=1}^m |\widehat{u}_k(\xi)|^2. \quad (2.11)$$

Lemma 2.5. *The following inequality holds:*

$$|\nabla F_S(\xi)| \leq 2(2\pi)^{-n} (n(n - 1))^{1/2} \sqrt{|\Omega| I}. \quad (2.12)$$

Proof. The proof is similar to that in [11] for the Laplacian. We have

$$\partial_j \widehat{u}_k^l(\xi) = -(2\pi)^{-n/2} i \int_{\Omega} u_k^l(x) x_j e^{-i\xi x} dx.$$

Since the family $\{u_k^l\}_{k=1}^m$ is subothonormal, by Lemma 2.3 we have

$$\sum_{k=1}^m |\partial_j \widehat{u}_k^l(\xi)|^2 \leq (2\pi)^{-n} \int_{\Omega} x_j^2 dx$$

and

$$\sum_{k=1}^m |\nabla \widehat{u}_k(\xi)|^2 \leq (2\pi)^{-n} n \int_{\Omega} x^2 dx = (2\pi)^{-n} n I(\Omega), \quad I(\Omega) = \int_{\Omega} x^2 dx.$$

Next, using (2.9) we obtain

$$|\nabla F_S(\xi)| \leq 2 \left(\sum_{k=1}^m |\widehat{u}_k(\xi)|^2 \right)^{1/2} \left(\sum_{k=1}^m |\nabla \widehat{u}_k(\xi)|^2 \right)^{1/2} \leq 2(2\pi)^{-n} (n(n-1))^{1/2} \sqrt{|\Omega|I}.$$

□

If, in addition, the orthonormal family $\{u_k\}_{k=1}^m$ belongs to $V \subseteq \{u \in \mathbf{H}_0^1(\Omega), \operatorname{div} u = 0\}$, then, by the Plancherel theorem, the function $F_S(\xi)$ defined in (2.11) satisfies

$$\begin{aligned} 0 &\leq F_S(\xi) \leq M_S = (2\pi)^{-n} (n-1) |\Omega|; \\ |\nabla F_S(\xi)| &\leq L_S = 2(2\pi)^{-n} (n(n-1))^{1/2} \sqrt{|\Omega|I}; \\ \int F_S(\xi) d\xi &= m; \\ \int |\xi|^2 F_S(\xi) d\xi &= \sum_{k=1}^m \|\nabla u_k\|^2. \end{aligned} \tag{2.13}$$

In the case of the Laplace operator, that is, for an orthonormal family $\{\varphi_k\}_{k=1}^m \in H_0^1(\Omega)$ the corresponding function $F_L(\xi) = \sum_{k=1}^m |\widehat{\varphi}_k(\xi)|^2$ satisfies [10], [11]

$$\begin{aligned} 0 &\leq F_L(\xi) \leq M_L = (2\pi)^{-n} |\Omega|; \\ |\nabla F_L(\xi)| &\leq L_L = 2(2\pi)^{-n} \sqrt{|\Omega|I}; \\ \int F_L(\xi) d\xi &= m; \\ \int |\xi|^2 F_L(\xi) d\xi &= \sum_{k=1}^m \|\nabla \varphi_k\|^2. \end{aligned} \tag{2.14}$$

3. MINIMIZATION PROBLEM

There is not much difference now between the Laplace and the Stokes operators, and the problem of lower bounds for the eigenvalues reduces to the problem of finding $\Sigma_{M,L}(m)$ defined in the minimization problem (1.8).

We consider the symmetric-decreasing rearrangement $F^*(\xi)$ of the $F(\xi)$. It is well known (see, for example, [13]) that $0 \leq F^*(\xi) \leq M$, $\int F^*(\xi) d\xi = \int F(\xi) d\xi = m$ and, in addition, $|\nabla F^*(\xi)| \leq \operatorname{ess\,sup} |\nabla F(\xi)|$. Also,

$$\int |\xi|^2 F(\xi) d\xi \geq \int |\xi|^2 F^*(\xi) d\xi. \tag{3.1}$$

This inequality follows from the Hardy–Littlewood inequality

$$\int G(\xi)F(\xi)d\xi \leq \int G^*(\xi)F^*(\xi)d\xi,$$

where $G(\xi) = G^*(\xi) = R^2 - |\xi|^2$ and without loss of generality we assume that the ball B_R contains the supports of F and F^* .

Thus, we obtain a one-dimensional problem equivalent to (1.8):

$$\begin{aligned} \sigma_n \int_0^\infty r^{n+1} F(r) dr &\rightarrow \inf =: \Sigma_{M,L}(m), \\ 0 \leq F(r) \leq M, \quad \sigma_n \int_0^\infty r^{n-1} F(r) dr &= m, \quad -F'(r) \leq L, \end{aligned} \quad (3.2)$$

where $F(r)$ is decreasing and without loss of generality we assume that F is absolutely continuous.

We consider the function $\Phi_s(r)$ shown in Fig. 2:

$$\Phi_s(r) = \begin{cases} M, & \text{for } 0 \leq r \leq s; \\ M - Lr, & \text{for } s \leq r \leq s + M/L; \\ 0, & \text{for } s + M/L \leq r. \end{cases} \quad (3.3)$$

Lemma 3.1. *Suppose that $\int_0^\infty r^\alpha \Phi_s(r) dr = m^*$ and $\beta \geq \alpha$. Then for any decreasing absolutely continuous function F satisfying the conditions*

$$0 \leq F \leq M, \quad \int_0^\infty r^\alpha F(r) dr = m^*, \quad -F' \leq L,$$

the following inequality holds:

$$\int_0^\infty r^\beta F(r) dr \geq \int_0^\infty r^\beta \Phi_s(r) dr. \quad (3.4)$$

Proof. If F is an admissible function and $F(s) = \Phi_s(s) (= M)$, then $F \equiv \Phi_s$. Hence for any admissible function F such that $F \neq \Phi_s$ (and, hence, $F(r_0) < M = \Phi_s(r_0)$ at some point r_0 , $0 \leq r_0 < s$), the graph of F intersects the graph of Φ_s to the right of r_0 at exactly one point with r -coordinate a , where a is in the region $s < a < s + M/L$. In other words, $F(r) \leq \Phi_s(r)$ for $0 \leq r \leq a$ and $F(r) \geq \Phi_s(r)$ for $a \leq r < \infty$. Therefore

$$\begin{aligned} \int_0^a r^\beta (\Phi_s(r) - F(r)) dr &\leq a^{\beta-\alpha} \int_0^a r^\alpha (\Phi_s(r) - F(r)) dr = \\ a^{\beta-\alpha} \int_a^\infty r^\alpha (F(r) - \Phi_s(r)) dr &\leq \int_a^\infty r^\beta (F(r) - \Phi_s(r)) dr, \end{aligned}$$

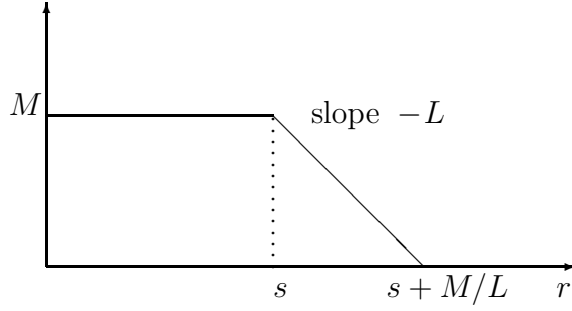
where the functions under the integral signs are non-negative. □

Lemma 3.2. *By a straight forward calculation*

$$\int_0^\infty r^\gamma \Phi_s(r) dr = \frac{M^{\gamma+2}}{(\gamma+1)(\gamma+2)L^{\gamma+1}} ((t+1)^{\gamma+2} - t^{\gamma+2}), \quad s = \frac{tM}{L}. \quad (3.5)$$

Combining the above results we see that the minimizing function is given by (3.3) and the second condition in (3.2) becomes $\sigma_n \int_0^\infty r^{n-1} \Phi_s(r) dr = m$, which in view of (3.5) gives the equation for t (and s):

$$(t+1)^{n+1} - t^{n+1} = m \frac{n(n+1)L^n}{\sigma_n M^{n+1}} = m \frac{(n+1)L^n}{\omega_n M^{n+1}} =: m_*. \quad (3.6)$$


 FIGURE 2. Minimizer $\Phi_s(|\xi|)$

It will be shown (see (3.11)) that for $m \geq 1$ the right-hand side in (3.6) is greater than 1. Since the left-hand side is a polynomial of order n (with positive coefficients) monotonely increasing from 1 to ∞ on \mathbb{R}^+ , the equation (3.6) has a unique solution $t = t(m_*) \geq 0$. Using (3.5) this time with $\gamma = n + 1$ we find the solution of (1.8), that is, $\Sigma_{M,L}(m)$. In other words, we have just proved the following result.

Proposition 3.1. *The solution of the minimization problem (1.8) is given by*

$$\Sigma_{M,L}(m) = \frac{\sigma_n M^{n+3}}{(n+2)(n+3)L^{n+2}} ((t(m_*) + 1)^{n+3} - t(m_*)^{n+3}), \quad (3.7)$$

where $t(m_*)$ is the unique positive root of the equation (3.6).

Remark 3.1. The shape of the minimizer (3.3) was found in [7]. We use it here to find the exact solution (3.7) of the minimization problem (1.8).

We give explicit expressions for $\Sigma_{M,L}(m)$ (and thereby explicit lower bounds for sums of eigenvalues of the Laplace and Stokes operators) for the dimension $n = 2, 3, 4$ in §4. Meanwhile we obtain the asymptotic expansion for $\Sigma_{M,L}(m)$ valid for all dimensions n .

First, it is convenient to write the right-hand side in (3.6) in the form

$$(t+1)^{n+1} - t^{n+1} = (\eta + 1/2)^{n+1} - (\eta - 1/2)^{n+1}, \quad \eta = t + 1/2, \quad (3.8)$$

since this substitution kills half of the coefficients in the explicit expression for the polynomial. Then the equation (3.8) takes the form

$$(n+1) \left(\eta^n + \frac{n(n-1)}{24} \eta^{n-2} + \frac{n(n-1)(n-2)(n-3)}{1920} \eta^{n-4} + \dots \right) = m_*.$$

The unique positive root $\eta(m_*)$ of this equation has the asymptotic expansion

$$\eta(m_*) = \left(\frac{m_*}{n+1} \right)^{1/n} - \frac{n-1}{24} \left(\frac{m_*}{n+1} \right)^{-1/n} + \frac{(n-1)(n-3)(2n-1)}{5760} \left(\frac{m_*}{n+1} \right)^{-3/n} + \dots \quad (3.9)$$

The first term here is obvious, the second and the third terms can be found in the standard way. Therefore substituting (3.9) into the second factor in (3.7) we obtain

$$\begin{aligned} (t(m_*) + 1)^{n+3} - t(m_*)^{n+3} &= (\eta(m_*) + 1/2)^{n+3} - (\eta(m_*) - 1/2)^{n+3} = \\ (n+3) &\left[\left(\frac{m_*}{n+1} \right)^{1+2/n} + \frac{(n+2)}{12} \frac{m_*}{n+1} - \frac{(n-1)(n+2)(3n+2)}{1440} \left(\frac{m_*}{n+1} \right)^{1-2/n} + \dots \right], \end{aligned} \quad (3.10)$$

and then (3.7) along with the expression for m_* in (3.6) finally gives (1.10).

Proof of Theorem 1.1. The difference between the Laplace and Stokes operators is now only in the definition of M and L and we consider the case of the Stokes operator. Since

$$\sum_{k=1}^m \|\nabla u_k\|^2 = \int |\xi|^2 F_S(\xi) d\xi \geq \Sigma_{M,L}(m),$$

it remains to substitute into (1.10) M_S and L_S from (2.13). This gives that $\sum_{k=1}^m \|\nabla u_k\|^2 \geq$ r. h. s. of (1.12) and inequality (1.12) follows by taking the first normalized eigenvectors of the Stokes problem for the u_k 's. The proof of (1.11) is totally similar. \square

We conclude this section by checking that both for the Laplace and Stokes operators $m_* \geq 1$, that is,

$$\frac{(n+1)L^n}{\omega_n M^{n+1}} \geq 1. \quad (3.11)$$

(Geometrically this means that Φ_s always has a horizontal part.) This follows from the inequality

$$I = \int_{\Omega} |x|^2 dx \geq \frac{n|\Omega|^{1+2/n}}{(n+2)\omega_n^{2/n}}, \quad (3.12)$$

which, in turn, is (3.1) with F being the characteristic function of Ω . In fact, (3.12) and the formulas for M and L give much more than (3.11):

$$m_* \geq m_0^L = \frac{(n+1)(4\pi)^n}{\omega_n^2} \left(\frac{n}{n+2} \right)^{n/2}, \quad m_* \geq m_0^S = \frac{(n+1)(4\pi)^n}{(n-1)\omega_n^2} \left(\frac{n^2}{(n-1)(n+2)} \right)^{n/2} \quad (3.13)$$

for the Laplace and Stokes operators, respectively, in the sense that the right-hand sides in (3.13) tend to infinity as $n \rightarrow \infty$.

4. LOWER BOUNDS FOR THE LAPLACE AND STOKES OPERATORS FOR $n = 2, 3, 4$

The case $n = 2$. The two-dimensional case is the simplest and the results are the most complete.

Lemma 4.1. *In the two-dimensional case*

$$\Sigma_{M,L}(m) = \frac{1}{2\pi M} m^2 + \frac{M^2}{6L^2} m - \frac{\pi M^5}{90L^4}. \quad (4.1)$$

Proof. In view of (3.7) we only need to calculate the last factor there. The positive root $t(m_*)$ of the equation (3.6) $_{n=2}$, which is the quadratic equation $(t+1)^3 - t^3 = m_*$, is

$$t(m_*) = \sqrt{\frac{m_*}{3} - \frac{1}{12}} - \frac{1}{2} \quad (4.2)$$

and using (3.8) we obtain

$$(t(m_*) + 1)^5 - t(m_*)^5 = \frac{5}{9} m_*^2 + \frac{5}{9} m_* - \frac{1}{9}.$$

The rest is a direct substitution. We note that $\Sigma_{M,L}(m) = \Sigma_0(m)_{n=2}$, see (1.10). \square

Theorem 4.1. *For $n = 2$ the eigenvalues of the Laplace and Stokes operators satisfy*

$$\sum_{k=1}^m \mu_k \geq \frac{2\pi}{|\Omega|} m^2 + \frac{1}{24} \frac{|\Omega|}{I} m \left(1 - \frac{1}{120m} \right) \geq \frac{2\pi}{|\Omega|} m^2 + \frac{1}{24} \frac{119}{120} \frac{|\Omega|}{I} m, \quad (4.3)$$

$$\sum_{k=1}^m \lambda_k \geq \frac{2\pi}{|\Omega|} m^2 + \frac{1}{48} \frac{|\Omega|}{I} m \left(1 - \frac{1}{240m} \right) \geq \frac{2\pi}{|\Omega|} m^2 + \frac{1}{48} \frac{239}{240} \frac{|\Omega|}{I} m. \quad (4.4)$$

Proof. We consider (4.3). In view of (2.14) we have $M = M_L = (2\pi)^{-2}|\Omega|$ and $L = L_L = 2(2\pi)^{-2}\sqrt{|\Omega|I}$, therefore (4.1) gives for the Laplacian

$$\sum_{k=1}^m \mu_k \geq \Sigma_{M,L}(m) = \frac{2\pi}{|\Omega|} m^2 + \frac{1}{24} \frac{|\Omega|}{I} m - \frac{1}{90 \cdot 2^6 \pi} \frac{|\Omega|^3}{I^2} \geq \frac{2\pi}{|\Omega|} m^2 + \frac{1}{24} \frac{|\Omega|}{I} m - \frac{1}{90 \cdot 2^5} \frac{|\Omega|}{I},$$

where the last inequality follows from (3.12): $|\Omega|^2/I \leq 2\pi$. The proof (4.4) is similar: $M = M_S = (2\pi)^{-2}|\Omega|$, $L = L_S = 2(2\pi)^{-2}\sqrt{2}\sqrt{|\Omega|I}$ and by (4.1)

$$\sum_{k=1}^m \lambda_k \geq \Sigma_{M,L}(m) = \frac{2\pi}{|\Omega|} m^2 + \frac{1}{48} \frac{|\Omega|}{I} m - \frac{1}{90 \cdot 2^8 \pi} \frac{|\Omega|^3}{I^2} \geq \frac{2\pi}{|\Omega|} m^2 + \frac{1}{48} \frac{|\Omega|}{I} m - \frac{1}{90 \cdot 2^7} \frac{|\Omega|}{I}.$$

The proof of this theorem (which is Theorem 1.2_{n=2}) is complete. \square

The case $n = 4$.

Lemma 4.2. *In the four-dimensional case*

$$\Sigma_{M,L}(m) \geq \frac{8\sqrt{2}}{3\pi M^{1/2}} m^{3/2} + \frac{1}{3} \cdot \beta \frac{M^2}{L^2} m, \quad (4.5)$$

where $\beta = \beta_4^L = 0.983$ for the Laplace operator and $\beta = \beta_4^S = 0.978$ for the Stokes operator.

Proof. The positive root $t(m_*)$ of the equation (3.6)_{n=4} (which is biquadratic with respect to $\eta = t + 1/2$) is

$$t(m_*) = \sqrt{\sqrt{20m_* + 5}/10 - 1/4 - 1/2}$$

and with the help of (3.8) we find that

$$\begin{aligned} \sigma(m_*) &:= (t(m_*) + 1)^7 - t(m_*)^7 = (7/50)(m_*\sqrt{20m_* + 5} + 5m_* - \sqrt{20m_* + 5} + 15/7) > \\ &\frac{7}{50} \left(2\sqrt{5}m_*^{3/2} + 5m_* - \frac{7\sqrt{5}}{4}m_*^{1/2} + \frac{15}{7} - \frac{17\sqrt{5}}{64}m_*^{-1/2} \right) > \frac{7\sqrt{5}}{25}m_*^{3/2} + \frac{7}{10}m_* - \frac{49\sqrt{5}}{200}m_*^{1/2}, \end{aligned}$$

where we used the inequality $1 + x/2 - x^2/8 < \sqrt{1+x} < 1 + x/2$ and the fact that $m_* \geq 1$. We observe that the three terms on the right here are as in (3.10)_{n=4} so that $\Sigma_{M,L}(m) > \Sigma_0(m)_{n=4}$, see (1.10).

We now take advantage of the fact that m_* is large, namely, $m_* \geq m_0^L = (5/9)2^{12} = 2275.5\dots$ and $m_* \geq m_0^S = 5 \cdot 2^{16}/3^5 = 1348.7\dots$, respectively, (see (3.13)). The smallest constant $\alpha > 0$ such that

$$\alpha m_* \geq \frac{49\sqrt{5}}{200}m_*^{1/2}, \quad m_* \in [m_0, \infty)$$

clearly is $\alpha_0 = (49\sqrt{5}/200)m_0^{-1/2}$. For the Laplace operator $\alpha_0^L = (49\sqrt{5}/200)(m_0^L)^{-1/2} = 0.01148\dots$, while for the Stokes operator $\alpha_0^S = 0.01491\dots$. Hence

$$\sigma(m_*) > \frac{7\sqrt{5}}{25}m_*^{3/2} + \frac{7}{10}\beta m_*, \quad \beta = 1 - \frac{10}{7}\alpha,$$

where $\beta^L = 0.9835\dots$ and $\beta^S = 0.9786\dots$, respectively, and (4.5) follows by going over from m_* to m (see (3.6), (3.7), (1.10)). \square

Proof of Theorem 1.2_{n=4}. We substitute the expressions for M and L into (4.5) and get the result. \square

The case $n = 3$.

Lemma 4.3. *In the tree-dimensional case*

$$\Sigma_{M,L}(m) \geq \frac{3}{5} \left(\frac{3}{4\pi M} \right)^{2/3} m^{5/3} + \frac{1}{4} \cdot \beta \frac{M^2}{L^2} m, \quad (4.6)$$

where $\beta = \beta_3^L = 0.9869$ and $\beta = \beta_3^S = 0.9861$ for the Laplace and Stokes operators, respectively.

Proof. The unique positive root $t(m_*)$ of the cubic equation $(3.6)_{n=3}$ is given by Cardano's formula (in which all the roots are taken positive)

$$t(m_*) = \frac{1}{2} \left(m_* + \sqrt{m_*^2 + \frac{1}{27}} \right)^{1/3} - \frac{1}{2} \left(-m_* + \sqrt{m_*^2 + \frac{1}{27}} \right)^{1/3} - \frac{1}{2}.$$

By a direct substitution using (3.8) we have

$$\begin{aligned} \sigma(m_*) &:= (t(m_*) + 1)^6 - t(m_*)^6 = \\ &\frac{1}{48} (3\sqrt{3+81m_*^2} + 27m_*)^{2/3} (11m_* - \sqrt{3+81m_*^2}) + \\ &\frac{5}{8} m_* + \\ &\frac{1}{48} \left((3\sqrt{3+81m_*^2} - 27m_*)^{2/3} (11m_* + \sqrt{3+81m_*^2}) - 7(3\sqrt{3+81m_*^2} + 27m_*)^{1/3} \right) + \\ &\frac{7}{48} (3\sqrt{3+81m_*^2} - 27m_*)^{1/3}, \end{aligned}$$

where the four terms above are written in the order $m_*^{5/3}$, m_* , $m_*^{1/3}$, $m_*^{-1/3}$. We now obtain a lower bound for $\sigma(m_*)$. Using the inequality $\sqrt{1+x} < 1+x/2$ below we get that the first term is greater than

$$\frac{3 \cdot 2^{2/3}}{8} m_*^{5/3} - \frac{2^{2/3}}{32} m_*^{-1/3}.$$

The third term is equal to

$$-\frac{90m_* + 12\sqrt{3+81m_*^2}}{48(3\sqrt{3+81m_*^2} + 27m_*)^{2/3}} > -\frac{198m_* + 2/m_*}{48(54m_*)^{2/3}} = -\frac{11 \cdot 2^{1/3}}{48} m_*^{1/3} - \frac{2^{1/3}}{48 \cdot 9} m_*^{-5/3}.$$

The fourth term is equal to

$$\frac{7}{16(3\sqrt{3+81m_*^2} + 27m_*)^{1/3}} > \frac{7m_*^{-1/3}}{48 \cdot 2^{1/3}} \left(1 + \frac{1}{27m_*^2} \right)^{-1/3} > \frac{7 \cdot 2^{1/3}}{96} m_*^{-1/3},$$

since $m_* \geq 1$. Collecting these estimates we obtain

$$\sigma(m_*) > \frac{3 \cdot 2^{2/3}}{8} m_*^{5/3} + \frac{5}{8} m_* - \frac{11 \cdot 2^{1/3}}{48} m_*^{1/3}, \quad (4.7)$$

so that as for $n = 4$ we have $\Sigma_{M,L}(m) > \Sigma_0(m)_{n=3}$, see (1.10).

As in Lemma 4.2 we have from (3.13) that $m_* \geq m_0^L = (16 \cdot 27\pi/5)(3/5)^{1/2} = 210.2 \dots$ and $m_* \geq m_0^S = 72\pi(9/10)^{3/2} = 193.1 \dots$ for the Laplace and Stokes operators, respectively. Therefore the inequality

$$\alpha m_* - \frac{11 \cdot 2^{1/3}}{48} m_*^{1/3} \geq 0, \quad m_* \in [m_0, \infty)$$

is satisfied for all $\alpha \geq \alpha_0 = \frac{11 \cdot 2^{1/3}}{48} m_0^{-2/3}$. Hence for the Laplace operator $\alpha_0^L = 0.008165 \dots$, while for the Stokes operator $\alpha_0^S = 0.008641 \dots$. Hence

$$\sigma(m_*) > \frac{3 \cdot 2^{2/3}}{8} m_*^{5/3} + \frac{5}{8} \beta m_*, \quad \beta = 1 - \frac{8}{5} \alpha,$$

where $\beta^L = 0.9869 \dots$ and $\beta^S = 0.9861 \dots$, respectively, which proves (4.6) (see (1.10)). \square

Proof of Theorem 1.2_{n=3}. The proof immediately follows from (4.6). The proof of Theorem 1.2 is complete. \square

5. FURTHER EXAMPLES. DIRICHLET BI-LAPLACIAN

Other elliptic equations and systems with constant coefficients and Dirichlet boundary conditions can be treated quite similarly. We restrict ourselves to the Dirichlet bi-Laplacian:

$$\Delta^2 \varphi_k = \nu_k \varphi_k, \quad \varphi_k|_{\partial\Omega} = 0, \quad \frac{\varphi_k}{\partial n}|_{\partial\Omega} = 0. \quad (5.1)$$

We consider the L_2 -orthonormal family of eigenfunctions $\{\varphi_k\}_{k=1}^m \in H_0^2(\Omega)$. Then the function $F(\xi) = \sum_{k=1}^m |\widehat{\varphi}_k(\xi)|^2$ satisfies the same three conditions:

$$1) \ 0 \leq F(\xi) \leq M, \quad 2) \ |\nabla F(\xi)| \leq L, \quad 3) \ \int_{\mathbb{R}^2} F(\xi) d\xi = m, \quad (5.2)$$

where as before $M = (2\pi)^{-n} |\Omega|$ and $L = 2(2\pi)^{-n} \sqrt{|\Omega|}$. Since $\sum_{k=1}^m \nu_k = \int_{\mathbb{R}^n} |\xi|^4 F(\xi) d\xi$, we have to find the solution $\Sigma_{M,L}^4(m)$ of the minimization problem

$$\int_{\mathbb{R}^2} |\xi|^4 f(\xi) d\xi \rightarrow \inf =: \Sigma_{M,L}^4(m) \quad \text{under conditions (5.2),} \quad (5.3)$$

whose solution is found similarly to Proposition 3.1.

Proposition 5.1. *The solution of the minimization problem (5.3) is given by*

$$\Sigma_{M,L}^4(m) = \frac{\sigma_n M^{n+5}}{(n+4)(n+5)L^{n+4}} ((t(m_*) + 1)^{n+5} - t(m_*)^{n+5}), \quad (5.4)$$

where $t(m_*)$ is the unique positive root of the equation (3.6).

Proof. The minimizer (3.3) and the equation for s (3.6) are the same as before. It remains to calculate the integral $\int_{\mathbb{R}^n} |\xi|^4 \Phi_s(|\xi|) d\xi$ based on Lemma 3.2. \square

We restrict ourselves to the least technical two-dimensional case.

Lemma 5.1. *For $n = 2$ the exact solution $\Sigma_{M,L}^4(m)$ can be found explicitly:*

$$\Sigma_{M,L}^4(m) = \frac{1}{3\pi^2 M^2} m^3 + \frac{M}{3\pi L^2} m^2 - \frac{\pi M^7}{7 \cdot 3^4 L^6}.$$

Proof. As before the unique positive root $t(m_*)$ of the equation $(t+1)^3 - t^3 = m_*$ is given by (4.2): $t(m_*) = \sqrt{m_*/3 - 1/2} - 1/2$, and a direct substitution gives

$$(t(m_*) + 1)^7 - t(m_*)^7 = \frac{7}{27} m_*^3 + \frac{7}{9} m_*^2 - \frac{1}{27}. \quad (5.5)$$

It remains to substitute (5.5) into (5.4) with

$$m_* = m \frac{(n+1)L^n}{\omega_n M^{n+1}} \Big|_{n=2} = m \frac{3L^2}{\pi M^3}.$$

\square

Theorem 5.1. *For $n = 2$ the eigenvalues of the Dirichlet bi-Laplacian satisfy*

$$\sum_{k=1}^m \nu_k \geq \frac{16\pi^2}{3|\Omega|^2} m^3 + \frac{\pi}{3I} m^2 \left(1 - \frac{1}{7 \cdot 3^3 \cdot 2^6 m^2} \right) \geq \frac{16\pi^2}{3|\Omega|^2} m^3 + \frac{\pi}{3I} \frac{12095}{12096} m^2. \quad (5.6)$$

Proof. Similar to Theorem 4.1. □

Remark 5.1. The coefficient of the leading term m^3 in (5.6) is sharp.

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